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Local stability of limit cycles for MIMO relay feedback systems

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Abstract

This paper concerns with the local stability of limit cycles for decentralized relay feedback systems. It presents a sufficient condition for the local stability based on the well-known Poincare map method. The effectiveness of the presented result is illustrated by a numerical example.

Keywords: Decentralized relay feedback; Hysteresis; Limit cycles; Local stability

1. Introduction

Relay feedback has attracted considerable research attention for more than century [7]. Applications of relay systems range from stationary control of industrial processes to control of mobile objects. Recent advances are relay auto-tuning of PID controllers [2,10] and process identification and control [9]. A phenomenon of relay feedback systems is that a particular type of periodic motions, i.e., limit cycle, may occur in the trajectories. It is meaningful to determine the stability of a limit cycle since this property is a pre-requisite in engineering applications.

For single-input single-output plants with a single relay element, say, SISO relay feedback systems, exact method has been developed to analyze limit cycle behaviors, see [1,4] and references therein. Astrom [1] gives elegant criteria for the local stability of limit cycles by considering the linear approximation of the Poincare map. The global stability issue is studied in [4].

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Practically and theoretically, multi-input multi-output systems are more common and useful, see [8,9] for details. Recent advances also show the important applications of multi-input multi-output systems connected with decentralized relays which called MIMO relay feedback systems [8,9]. So far, there are few efforts devoted to the study of limit cycles for MIMO relay feedback systems. Palmor et al. [5,6] present a frequency domain based method for evaluating the periods and the stability of limit cycles in decentralized relay systems. The Z-transform technique is employed therein to convert the continuous decentralized relay system under a limit cycle to an equivalent fictitious sampled-data system with synchronous samplers. Then, regular sampled-data tools are applied to derive closed-form necessary conditions as well as stability conditions.

In this paper, we will revisit the local stability of limit cycles for decentralized relay systems through exact method. The analysis is state-space based and takes into the exact consideration of the Poincare map. The result is novel in the sense that it is to check the Schur stability of a certain matrix which is constructed by the original system parameters and the limit cycle parameters. The criterion can be viewed as a generalization of that given in [1] for SISO systems. This paper is organized as follows. In Section 2, the considered decentralized relay system and problem are formulated. Section 3 gives a closed formula for computing the period of the considered limit cycle. Section 4 presents a sufficient condition for the local stability of the limit cycle. A numerical example is given in Section 5 to illustrate our result. This paper is concluded in Section 6.

2. Problem formulation and preliminaries

The multi-input multi-output system considered in this paper is shown in Fig. 1. The linear plant is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) = [y_1(t), y_2(t), \ldots, y_m(t)]^T \in \mathbb{R}^m$ and $u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ are the state, output and control input, respectively; A, $B = [b_1, b_2, \ldots, b_m]$ and $C = [c_1^T, c_2^T, \ldots, c_m^T]^T$ are constant real matrices with $b_i, c_i^T \in \mathbb{R}^n$. The plant is under decentralized relay feedback:

$$u_i(t) = \begin{cases} u_{\beta_i} & \text{if } y_i(t) > \beta_i, \text{ or } y_i(t) \geqslant \alpha_i \text{ and } u_i(t_-) = u_{\beta_i}, \\ u_{\alpha_i} & \text{if } y_i(t) < \alpha_i, \text{ or } y_i(t) \leqslant \beta_i \text{ and } u_i(t_-) = u_{\alpha_i}, \end{cases} \quad i = 1, 2, \ldots, m, \tag{2}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ with $\alpha_i \leqslant \beta_i$ stand for the hysteresis; $u_{\alpha_i}, u_{\beta_i} \in \mathbb{R}$ and $u_{\alpha_i} \neq u_{\beta_i}$. We specify the initial value $u(0)$ as

$$u_i(0) \equiv \begin{cases} u_{\beta_i} & \text{if } y_i(0) > \alpha_i, \\ u_{\alpha_i} & \text{if } y_i(0) \leqslant \alpha_i, \end{cases} \quad i = 1, 2, \ldots, m. \tag{3}$$

We call (1)–(3) a decentralized relay feedback system and denote by Σ. Note that although system Σ appears to be linear, in fact it is not due to the nonlinear control inputs. Here, a solution $x(t)$ to system Σ is defined in the sense of Filippov [3], i.e., an absolutely continuous function $x(t)$ is called a solution to system Σ if it satisfies Eqs. (1)–(3) almost everywhere.

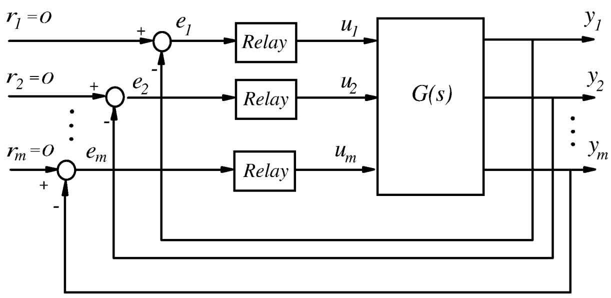

Fig. 1. Decentralized relay feedback systems.

Define the switching planes

$$\mathcal{S}_{\alpha_i} := \{\xi \in \mathbb{R}^n\colon\ c_i\xi = \alpha_i\}, \quad \mathcal{S}_{\beta_i} := \{\xi \in \mathbb{R}^n\colon\ c_i\xi = \beta_i\}, \quad i = 1, 2, \ldots, m. \tag{4}$$

Let

$$\mathcal{S}^+_{\alpha_i} := \{\xi \in \mathbb{R}^n\colon\ c_i\xi > \alpha_i\}, \quad \mathcal{S}^-_{\alpha_i} := \{\xi \in \mathbb{R}^n\colon\ c_i\xi < \alpha_i\}, \quad i = 1, 2, \ldots, m, \tag{5}$$

and let $\mathcal{S}^+_{\beta_i}$ and $\mathcal{S}^-_{\beta_i}$ be defined similarly. For a certain $i = 1, 2, \ldots, m$, if a trajectory of system Σ, evolving from $\mathcal{S}^+_{\beta_i}$ (respectively, $\mathcal{S}^-_{\alpha_i}$), traverses $\mathcal{S}_{\alpha_i}$ (respectively, $\mathcal{S}_{\beta_i}$) at x, then we will call the state x a *traversing point*. The time instant corresponding to the traversing point is called *switching instant*. It should be stressed that in our convention for $\alpha_i < \beta_i$, if a trajectory traverses $\mathcal{S}_{\alpha_i}$ at x from $\mathcal{S}^-_{\alpha_i}$ (respectively, traverses $\mathcal{S}_{\beta_i}$ at x from $\mathcal{S}^+_{\beta_i}$), the state x is not called a traversing point, since such traversing does not cause a switch in $u(t)$.

In this note, we will study the local stability of a certain type of limit cycles. The local stability means that all nearby trajectories converge to the limit cycle as time tends to infinity. A sufficient condition is given in terms of the spectral radius of a constructed matrix.

3. Determination of limit cycles

The starting point in the analysis is to assume that a limit cycle exists in system Σ. As in [5,6], we assume that the outputs from all the relays under the limit cycle are square waves with the same fundamental period, but with different phase shifts. Without loss of generality, assume that the m relays switch in the sequence of $u_1, u_2, \ldots, u_m$. In a more detail, the considered limit cycle is of the following form.

Form *I*. Each relay under the limit cycle switches two times within a fundamental period. The fundamental period is $T = \sum_{i=1}^m T_{\alpha_i} + \sum_{i=1}^m T_{\beta_i}$, where T_{α_i} and T_{β_i}, $i = 1, 2, \ldots, m-1$, are, respectively, the time durations for the trajectory of the limit cycle to move from the traversing point $x^*_{\alpha_i} \in \mathcal{S}_{\alpha_i}$ to the successive one $x^*_{\alpha_{i+1}} \in \mathcal{S}_{\alpha_{i+1}}$ and from the traversing point $x^*_{\beta_i} \in \mathcal{S}_{\beta_i}$ to the successive one $x^*_{\beta_{i+1}} \in \mathcal{S}_{\beta_{i+1}}$, and T_{α_m} and T_{β_m} are, respectively, from $x^*_{\alpha_m}$ to $x^*_{\beta_1}$ and from $x^*_{\beta_m}$ to $x^*_{\alpha_1}$.

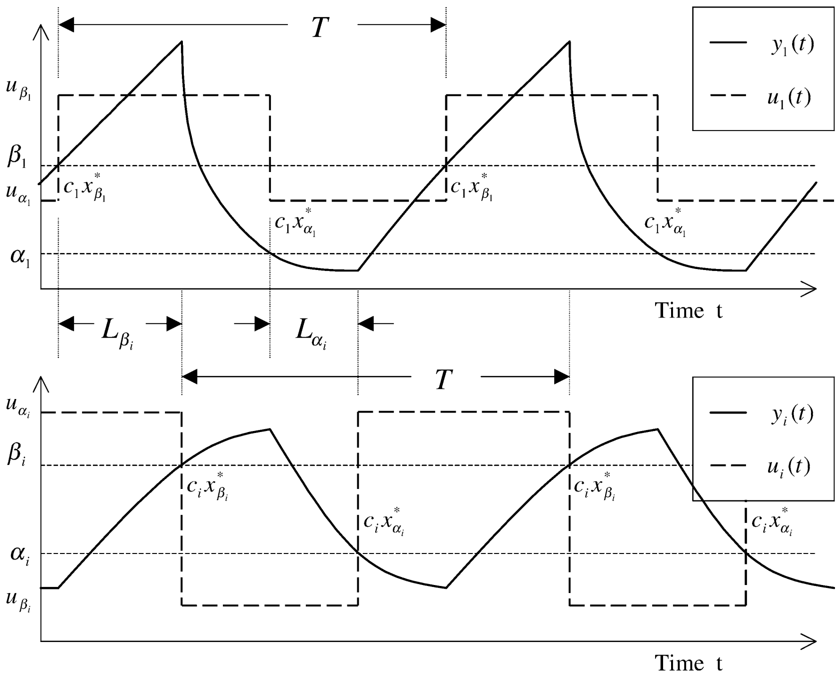

Fig. 2. Limit cycle behavior.

For illustration, see Fig. 2, where $L_{\alpha_i} = \sum_{j=1}^{i-1} T_{\alpha_j}$, $i = 2, 3, \ldots, m$, is the time difference between the first switching on $\mathcal{S}_{\alpha_1}$ and the ith switching on $\mathcal{S}_{\alpha_i}$, while $L_{\beta_i} = \sum_{j=1}^{i-1} T_{\beta_j}$, $i = 2, 3, \ldots, m$, is that between the first switching on $\mathcal{S}_{\beta_1}$ and the ith switching on $\mathcal{S}_{\beta_i}$. As in the literature, we assume by default that the considered limit cycle is nontangent with the switching planes at the switching instants.

In what follows, denote

$$\tilde{u}_{\alpha_i} := \begin{bmatrix} u_{\alpha_1} \\ \vdots \\ u_{\alpha_i} \\ u_{\beta_{i+1}} \\ \vdots \\ u_{\beta_m} \end{bmatrix}, \quad \tilde{u}_{\beta_i} := \begin{bmatrix} u_{\beta_1} \\ \vdots \\ u_{\beta_i} \\ u_{\alpha_{i+1}} \\ \vdots \\ u_{\alpha_m} \end{bmatrix}, \quad i = 1, 2, \ldots, m. \tag{6}$$

For convenience, we use the sum, $\sum_{k}^{k-1}$, where k is a natural number, to denote a zero term.

The following is a necessary condition for the existence of the limit cycle in Form I.

Proposition 3.1. *Suppose that the matrix, $I - e^{AT}$, is invertible (i.e., $\lambda(A) \neq j2l\pi/T$ for any integer l). If there exists a limit cycle of Form* I *in system Σ, then it holds that*

$$c_k x^*_{\alpha_k} = \alpha_k, \quad c_k x^*_{\beta_k} = \beta_k, \quad k = 1, 2, \ldots, m, \tag{7}$$

where

$$
\begin{aligned}
x_{\alpha_k}^* = (I - e^{AT})^{-1}\Biggl(&\sum_{j=k}^{m} \int_0^{T_{\beta_j}} e^{A(\sum_{i=1}^{k-1} T_{\alpha_i} + \sum_{i=j}^{m} T_{\beta_i} - s)} B\tilde{u}_{\beta_j}\, ds \\
&+ \sum_{j=1}^{k-1} \int_0^{T_{\alpha_j}} e^{A(\sum_{i=j}^{k-1} T_{\alpha_i} - s)} B\tilde{u}_{\alpha_j}\, ds \\
&+ \sum_{j=k}^{m} \int_0^{T_{\alpha_j}} e^{A(\sum_{i=1}^{m} T_{\beta_i} + \sum_{i=1}^{k-1} T_{\alpha_i} + \sum_{i=j}^{m} T_{\alpha_i} - s)} B\tilde{u}_{\alpha_j}\, ds \\
&+ \sum_{j=1}^{k-1} \int_0^{T_{\beta_j}} e^{A(\sum_{i=1}^{k-1} T_{\alpha_i} + \sum_{i=j}^{m} T_{\beta_i} - s)} B\tilde{u}_{\beta_j}\, ds \Biggr), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
x_{\beta_k}^* = (I - e^{AT})^{-1}\Biggl(&\sum_{j=k}^{m} \int_0^{T_{\alpha_j}} e^{A(\sum_{i=1}^{k-1} T_{\beta_i} + \sum_{i=j}^{m} T_{\alpha_i} - s)} B\tilde{u}_{\alpha_j}\, ds \\
&+ \sum_{j=1}^{k-1} \int_0^{T_{\beta_j}} e^{A(\sum_{i=j}^{k-1} T_{\beta_i} - s)} B\tilde{u}_{\beta_j}\, ds \\
&+ \sum_{j=k}^{m} \int_0^{T_{\beta_j}} e^{A(\sum_{i=1}^{m} T_{\alpha_i} + \sum_{i=1}^{k-1} T_{\beta_i} + \sum_{i=j}^{m} T_{\beta_i} - s)} B\tilde{u}_{\beta_j}\, ds \\
&+ \sum_{j=1}^{k-1} \int_0^{T_{\alpha_j}} e^{A(\sum_{i=1}^{k-1} T_{\beta_i} + \sum_{i=j}^{m} T_{\alpha_i} - s)} B\tilde{u}_{\alpha_j}\, ds \Biggr). \qquad (9)
\end{aligned}
$$

Proof. If there exists a limit cycle of Form I, it should satisfy

$$c_k x_{\alpha_k}^* = \alpha_k, \quad c_k x_{\beta_k}^* = \beta_k, \quad k = 1, 2, \ldots, m, \qquad (10)$$

and

$$x_{\alpha_1}^* = e^{AT_{\beta_m}} x_{\beta_m}^* + \int_0^{T_{\beta_m}} e^{A(T_{\beta_m} - s)} B\tilde{u}_{\beta_m}\, ds,$$

$$x_{\alpha_k}^* = e^{AT_{\alpha_{k-1}}} x_{\alpha_{k-1}}^* + \int_0^{T_{\alpha_{k-1}}} e^{A(T_{\alpha_{k-1}} - s)} B\tilde{u}_{\alpha_{k-1}}\, ds, \quad k = 2, \ldots, m,$$

$$x_{\beta_1}^* = e^{AT_{\alpha_m}} x_{\alpha_m}^* + \int_0^{T_{\alpha_m}} e^{A(T_{\alpha_m}-s)} B\tilde{u}_{\alpha_m}\, ds,$$

$$x_{\beta_k}^* = e^{AT_{\beta_{k-1}}} x_{\beta_{k-1}}^* + \int_0^{T_{\beta_{k-1}}} e^{A(T_{\beta_{k-1}}-s)} B\tilde{u}_{\beta_{k-1}}\, ds, \quad k = 2, \ldots, m. \tag{11}$$

From (11), we obtain for $k = 1, 2, \ldots, m$ that

$$x_{\alpha_k}^* = e^{A(\sum_{i=1}^{k-1} T_{\alpha_i} + \sum_{i=k}^{m} T_{\beta_i})} x_{\beta_k}^* + \sum_{j=k}^{m} \int_0^{T_{\beta_j}} e^{A(\sum_{i=1}^{k-1} T_{\alpha_i} + \sum_{i=j}^{m} T_{\beta_i} - s)} B\tilde{u}_{\beta_j}\, ds$$
$$+ \sum_{j=1}^{k-1} \int_0^{T_{\alpha_j}} e^{A(\sum_{i=j}^{k-1} T_{\alpha_i} - s)} B\tilde{u}_{\alpha_j}\, ds, \tag{12}$$

$$x_{\beta_k}^* = e^{A(\sum_{i=1}^{k-1} T_{\beta_i} + \sum_{i=k}^{m} T_{\alpha_i})} x_{\alpha_k}^* + \sum_{j=k}^{m} \int_0^{T_{\alpha_j}} e^{A(\sum_{i=1}^{k-1} T_{\beta_i} + \sum_{i=j}^{m} T_{\alpha_i} - s)} B\tilde{u}_{\alpha_j}\, ds$$
$$+ \sum_{j=1}^{k-1} \int_0^{T_{\beta_j}} e^{A(\sum_{i=j}^{k-1} T_{\beta_i} - s)} B\tilde{u}_{\beta_j}\, ds. \tag{13}$$

Note that $T = \sum_{i=1}^{m} T_{\alpha_i} + \sum_{i=1}^{m} T_{\beta_i}$. Left multiplying (13) by $e^{A(\sum_{i=1}^{k-1} T_{\alpha_i} + \sum_{i=k}^{m} T_{\beta_i})}$ and combining with (12) yield (8) while left multiplying (12) by $e^{A(\sum_{i=1}^{k-1} T_{\beta_i} + \sum_{i=k}^{m} T_{\alpha_i})}$ and combining with (13) yield (9). This proves the proposition. □

Remark 3.1. Equation (7) gives a closed form for solving the parameters of the fundamental period, T_{α_i} and T_{β_i}, $i = 1, 2, \ldots, m$. Numerical procedure has to be used. Once the solutions correspond to a limit cycle, the parameters of the traversing points are obtained as in (8) and (9). It is easy to see that for the special case when the system is SISO one ($m = 1$) with $\alpha_1 + \beta_1 = 0$, Proposition 3.1 reduces to Theorem 5.1 in [1] or Theorem 2.1 in [1] for symmetric limit cycle and $u_{\alpha_1} + u_{\beta_1} = 0$.

4. Criterion for local stability of limit cycles

The main result in this section is as follows.

Theorem 4.1. *Suppose there exists a limit cycle of Form* I *in system Σ. Then the limit cycle is locally stable if*

$$\rho\left(\prod_{j=1}^{m} W_{\beta_j} \prod_{i=1}^{m} W_{\alpha_i}\right) < 1, \tag{14}$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix, and

$$
\begin{aligned}
W_{\alpha_i} &= \left(I - \frac{(Ax^*_{\alpha_{i+1}} + B\tilde{u}_{\alpha_i})c_{i+1}}{c_{i+1}(Ax^*_{\alpha_{i+1}} + B\tilde{u}_{\alpha_i})}\right)e^{AT_{\alpha_i}}, \quad i = 1, 2, \ldots, m-1, \\
W_{\alpha_m} &= \left(I - \frac{(Ax^*_{\beta_1} + B\tilde{u}_{\alpha_m})c_1}{c_1(Ax^*_{\beta_1} + B\tilde{u}_{\alpha_m})}\right)e^{AT_{\alpha_m}}, \\
W_{\beta_j} &= \left(I - \frac{(Ax^*_{\beta_{j+1}} + B\tilde{u}_{\beta_j})c_{j+1}}{c_{j+1}(Ax^*_{\beta_{j+1}} + B\tilde{u}_{\beta_j})}\right)e^{AT_{\beta_j}}, \quad j = 1, 2, \ldots, m-1, \\
W_{\beta_m} &= \left(I - \frac{(Ax^*_{\alpha_1} + B\tilde{u}_{\beta_m})c_1}{c_1(Ax^*_{\alpha_1} + B\tilde{u}_{\beta_m})}\right)e^{AT_{\beta_m}}.
\end{aligned} \tag{15}
$$

Proof. Consider the limit cycle in Form I. Without loss of generality, we let $t_0 = 0$ correspond to the time instant when the relay switch from $\tilde{u}_{\beta_{m-1}}$ to $\tilde{u}_{\beta_m}$. This means that the initial point of the limit cycle is chosen to be $x_0^* = x^*_{\beta_m}$ and it holds that $c_i x_0^* > \alpha_i$, $i = 1, 2, \ldots, m$. Let

$$\epsilon_0 := \min_i \bigl\{ \bigl(c_i x_0^* - \alpha_i\bigr) \|c_i\|^{-1} \bigr\}. \tag{16}$$

Define the ϵ-neighborhood around x_0^* as

$$\mathcal{R}_\epsilon := \bigl\{\xi \in R^n\colon\ \bigl\|\xi - x_0^*\bigr\| < \epsilon\bigr\} = \bigl\{\xi \in R^n\colon\ \xi = x_0^* + \Delta,\ \Delta \in R^n,\ \|\Delta\| < \epsilon\bigr\}. \tag{17}$$

Then, any trajectory of system Σ starting at $x_0 = x_0^* + \Delta \in \mathcal{R}_{\epsilon_0}$ can evolve with $u(0) = \tilde{u}_{\beta_m}$. This is because $c_i x_0 = c_i(x_0^* + \Delta) > \alpha_i$, $i = 1, 2, \ldots, m$, for $\|\Delta\| < \epsilon_0$.

We now analyze the trajectory of $x(t)$ starting from a nearby point to x_0^*. By continuity, if ϵ ($\leqslant \epsilon_0$) is small enough, then any trajectory starting from $x_0 \in \mathcal{R}_\epsilon$ will traverse $\mathcal{S}_{\alpha_1}$ at a certain point $x^{(1)}_{\alpha_1}$. Moreover, $\|x^{(1)}_{\alpha_1} - x^*_{\alpha_1}\|$ can be made arbitrarily small by choosing ϵ, and thus $\|x_0 - x_0^*\|$, sufficiently small. With a similar way, by choosing x_0 close enough to x_0^*, the trajectory starting from x_0 will evolve close to the limit cycle (while the relay switches for the second, third, ..., $(2m+1)$th time) and return to traverse $\mathcal{S}_{\alpha_1}$ at another point $x^{(2)}_{\alpha_1}$. The Poincare map $P : \mathcal{R}_\epsilon \cap \mathcal{S}_{\alpha_1} \to \mathcal{S}_{\alpha_1}$ is defined as $P(x^{(1)}_{\alpha_1}) = x^{(2)}_{\alpha_1}$. Next, we compute the exact expression of P by relating $x^{(2)}_{\alpha_1} - x^*_{\alpha_1}$ to $x^{(1)}_{\alpha_1} - x^*_{\alpha_1}$.

Let the trajectory of $x(t)$ spend a time duration $T_{\alpha_1} + \delta^{(1)}_{\alpha_1}$ from $x^{(1)}_{\alpha_1} \in \mathcal{S}_{\alpha_1}$ to traverse $\mathcal{S}_{\alpha_2}$ at $x^{(1)}_{\alpha_2}$. Then,

$$
\begin{aligned}
x^{(1)}_{\alpha_2} &= e^{A(T_{\alpha_1} + \delta^{(1)}_{\alpha_1})}x^{(1)}_{\alpha_1} + \int_0^{T_{\alpha_1} + \delta^{(1)}_{\alpha_1}} e^{A(T_{\alpha_1} + \delta^{(1)}_{\alpha_1} - s)}B\tilde{u}_{\alpha_1}\,ds, \\
c_2 x^{(1)}_{\alpha_2} &= \alpha_2.
\end{aligned} \tag{18}
$$

Taking into account

$$x_{\alpha_2}^* = e^{AT_{\alpha_1}} x_{\alpha_1}^* + \int_0^{T_{\alpha_1}} e^{A(T_{\alpha_1}-s)} B\tilde{u}_{\alpha_1}\, ds,$$
$$c_2 x_{\alpha_2}^* = \alpha_2, \tag{19}$$

after some manipulations, we have

$$c_2 e^{A(T_{\alpha_1}+\delta_{\alpha_1}^{(1)})}\big(x_{\alpha_1}^{(1)} - x_{\alpha_1}^*\big) + c_2\big(e^{A\delta_{\alpha_1}^{(1)}} - I\big)x_{\alpha_2}^* + c_2 \int_0^{\delta_{\alpha_1}^{(1)}} e^{As} B\tilde{u}_{\alpha_1}\, ds = 0. \tag{20}$$

For $t \in \mathbb{R}$, define

$$f_{\alpha_1}(t) := c_2(e^{At} - I)x_{\alpha_2}^* + c_2 \int_0^t e^{As} B\tilde{u}_{\alpha_1}\, ds.$$

Then,

$$t^{-1} f_{\alpha_1}(t) \to c_2 A x_{\alpha_2}^* + c_2 B\tilde{u}_{\alpha_1} < 0 \quad \text{as } t \to 0.$$

By defining

$$t^{-1} f_{\alpha_1}(t)\big|_{t=0} := \lim_{t\to 0} t^{-1} f_{\alpha_1}(t),$$

there exists a scalar $r_{\alpha_1} > 0$ such that $t^{-1} f_{\alpha_1}(t) < 0$ is continuous on $t \in [-r_{\alpha_1}, r_{\alpha_1}]$. So, $\delta_{\alpha_1}^{(1)} f_{\alpha_1}^{-1}(\delta_{\alpha_1}^{(1)})$ is well defined by choosing small ϵ such that $|\delta_{\alpha_1}^{(1)}| \leqslant r_{\alpha_1}$. This enables us to get from (20) that

$$\delta_{\alpha_1}^{(1)} = -\delta_{\alpha_1}^{(1)} f_{\alpha_1}^{-1}\big(\delta_{\alpha_1}^{(1)}\big) c_2 e^{A(T_{\alpha_1}+\delta_{\alpha_1}^{(1)})}\big(x_{\alpha_1}^{(1)} - x_{\alpha_1}^*\big). \tag{21}$$

Using (18), (19) and (21), after simple deductions, we obtain

$$\begin{aligned}
x_{\alpha_2}^{(1)} - x_{\alpha_2}^* &= e^{A(T_{\alpha_1}+\delta_{\alpha_1}^{(1)})}\big(x_{\alpha_1}^{(1)} - x_{\alpha_1}^*\big) + \frac{(e^{A\delta_{\alpha_1}^{(1)}} - I)x_{\alpha_2}^* + \int_0^{\delta_{\alpha_1}^{(1)}} e^{As} B\tilde{u}_{\alpha_1}\, ds}{\delta_{\alpha_1}^{(1)}} \delta_{\alpha_1}^{(1)} \\
&= \left(I - \frac{((e^{A\delta_{\alpha_1}^{(1)}} - I)x_{\alpha_2}^* + \int_0^{\delta_{\alpha_1}^{(1)}} e^{As} B\tilde{u}_{\alpha_1}\, ds)c_2}{c_2((e^{A\delta_{\alpha_1}^{(1)}} - I)x_{\alpha_2}^* + \int_0^{\delta_{\alpha_1}^{(1)}} e^{As} B\tilde{u}_{\alpha_1}\, ds)}\right) \\
&\quad \times e^{A(T_{\alpha_1}+\delta_{\alpha_1}^{(1)})}\big(x_{\alpha_1}^{(1)} - x_{\alpha_1}^*\big).
\end{aligned} \tag{22}$$

The above shows the relation between $x_{\alpha_2}^{(1)} - x_{\alpha_2}^*$ and $x_{\alpha_1}^{(1)} - x_{\alpha_1}^*$. Similar deduction leads to the relation between $x_{\alpha_1}^{(2)} - x_{\alpha_1}^*$ and $x_{\alpha_1}^{(1)} - x_{\alpha_1}^*$, given by

$$x_{\alpha_1}^{(2)} - x_{\alpha_1}^* = \left(\prod_{j=1}^m W_{\beta_j}\big(\delta_{\beta_j}^{(1)}\big) \prod_{i=1}^m W_{\alpha_i}\big(\delta_{\alpha_i}^{(1)}\big)\right)\big(x_{\alpha_1}^{(1)} - x_{\alpha_1}^*\big), \tag{23}$$

where

$$
\begin{aligned}
&W_{\alpha_i}\big(\delta_{\alpha_i}^{(1)}\big) = \left(I - \frac{((e^{A\delta_{\alpha_i}^{(1)}} - I)x_{\alpha_{i+1}}^* + \int_0^{\delta_{\alpha_i}^{(1)}} e^{As}B\tilde{u}_{\alpha_i}\,ds)c_{i+1}}{c_{i+1}((e^{A\delta_{\alpha_i}^{(1)}} - I)x_{\alpha_{i+1}}^* + \int_0^{\delta_{\alpha_i}^{(1)}} e^{As}B\tilde{u}_{\alpha_i}\,ds)}\right)e^{A(T_{\alpha_i}+\delta_{\alpha_i}^{(1)})},\\
&\quad i = 1, 2, \ldots, m-1,\\
&W_{\alpha_m}\big(\delta_{\alpha_m}^{(1)}\big) = \left(I - \frac{((e^{A\delta_{\alpha_m}^{(1)}} - I)x_{\beta_1}^* + \int_0^{\delta_{\alpha_m}^{(1)}} e^{As}B\tilde{u}_{\alpha_m}\,ds)c_1}{c_1((e^{A\delta_{\alpha_m}^{(1)}} - I)x_{\beta_1}^* + \int_0^{\delta_{\alpha_m}^{(1)}} e^{As}B\tilde{u}_{\alpha_m}\,ds)}\right)e^{A(T_{\alpha_m}+\delta_{\alpha_m}^{(1)})},\\
&W_{\beta_j}\big(\delta_{\beta_j}^{(1)}\big) = \left(I - \frac{((e^{A\delta_{\beta_j}^{(1)}} - I)x_{\beta_{j+1}}^* + \int_0^{\delta_{\beta_j}^{(1)}} e^{As}B\tilde{u}_{\beta_j}\,ds)c_{j+1}}{c_{j+1}((e^{A\delta_{\beta_j}^{(1)}} - I)x_{\beta_{j+1}}^* + \int_0^{\delta_{\beta_j}^{(1)}} e^{As}B\tilde{u}_{\beta_j}\,ds)}\right)e^{A(T_{\beta_j}+\delta_{\beta_j}^{(1)})},\\
&\quad j = 1, 2, \ldots, m-1,\\
&W_{\beta_m}\big(\delta_{\beta_m}^{(1)}\big) = \left(I - \frac{((e^{A\delta_{\beta_m}^{(1)}} - I)x_{\alpha_1}^* + \int_0^{\delta_{\beta_m}^{(1)}} e^{As}B\tilde{u}_{\beta_m}\,ds)c_1}{c_1((e^{A\delta_{\beta_m}^{(1)}} - I)x_{\alpha_1}^* + \int_0^{\delta_{\beta_m}^{(1)}} e^{As}B\tilde{u}_{\beta_m}\,ds)}\right)e^{A(T_{\beta_m}+\delta_{\beta_m}^{(1)})}.
\end{aligned}
\tag{24}
$$

Furthermore, all the time differences, $\delta_{\alpha_k}^{(1)}$ and $\delta_{\beta_k}^{(1)}$, $k = 1, 2, \ldots, m$ (of the time durations for the two trajectories of $x(t)$ and the limit cycle to move from a traversing point to the successive one), can be arbitrarily close to zero by choosing ϵ sufficiently small.

Now, letting $\delta_{\alpha_k}^{(1)} \to 0$ and $\delta_{\beta_k}^{(1)} \to 0$, we see that $W_{\alpha_k}(\delta_{\alpha_k}^{(1)}) \to W_{\alpha_k}$ and $W_{\beta_k}(\delta_{\beta_k}^{(1)}) \to W_{\beta_k}$, $k = 1, 2, \ldots, m$. From theory of discrete-time systems, it can be shown that if the condition in (14) is satisfied, then there exists a scalar ϵ ($\leqslant \epsilon_0$) such that any trajectory starting from $\mathcal{R}_\epsilon$ will traverse $\mathcal{S}_{\alpha_1}$ consecutively. Moreover, the lth returned traversing point, $x_{\alpha_1}^{(l+1)}$, which relates to its former one $x_{\alpha_1}^{(l)}$ by a formula similar to that in (23), tends to $x_{\alpha_1}^*$ as the natural number l increases. This completes the proof of the theorem. □

Remark 4.1. The matrices in (15) satisfy that $c_1 W_{\alpha_m} = c_{i+1} W_{\alpha_i} = c_1 W_{\beta_m} = c_{j+1} W_{\beta_j} = 0$ for $i, j = 1, 2, \ldots, m-1$. So, the matrix in (14) always has a zero eigenvalue.

Remark 4.2. For the special case when the system is SISO one ($m = 1$) with $\alpha_1 + \beta_1 = 0$, Theorem 4.1 reduces to Theorem 5.2 in [1] or Theorem 3.1 in [1] for symmetric limit cycle and $u_{\alpha_1} + u_{\beta_1} = 0$.

Remark 4.3. The method in [5,6] is frequency-domain based and uses the Z-transform technique to convert the system to an equivalent fictitious sampled-data system with synchronous samplers, while our analysis is state-space based and the stability criterion in Theorem 4.1 is easy to apply. Moreover, the result in [5,6] is for symmetric limit cycles only while our result is also applicable to asymmetric limit cycles.

5. A numerical example

In this section, we give a numerical example to illustrate the use of our results.

Example 5.1. Consider system Σ with

$$A=\begin{bmatrix}-0.4 & 0 & 0.1\\ 0.5 & -0.5 & 0.1\\ 0.3 & 0 & -0.6\end{bmatrix},\qquad B=\begin{bmatrix}0.1 & 1\\ 1 & 0.2\\ 1 & 1\end{bmatrix},\qquad C=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},$$

$$\alpha_1=-0.3,\qquad \alpha_2=-0.1,\qquad \beta_1=0.2,\qquad \beta_2=0.1,$$

$$u_{\alpha_1}=-1,\qquad u_{\alpha_2}=3,\qquad u_{\beta_1}=1.5,\qquad u_{\beta_2}=-1.$$

The system has a limit cycle with the fundamental period $T=2.5482$ which meets Form I. The parameters of the period and the four traversing points are computed to be

$$T_{\alpha_1}=0.7894,\qquad T_{\alpha_2}=0.3897,\qquad T_{\beta_1}=0.1680,\qquad T_{\beta_2}=1.2011,$$

$$x^*_{\alpha_1}=\begin{bmatrix}-0.3000\\ 1.3486\\ 0.7932\end{bmatrix},\qquad x^*_{\alpha_2}=\begin{bmatrix}-0.9753\\ -0.1000\\ -0.8904\end{bmatrix},$$

$$x^*_{\beta_1}=\begin{bmatrix}0.2000\\ -0.3049\\ -0.0468\end{bmatrix},\qquad x^*_{\beta_2}=\begin{bmatrix}0.7001\\ 0.1000\\ 0.6986\end{bmatrix}.$$

Since the limit cycle is not symmetric, the result in [5,6] is not applicable. Now, we use Theorem 4.1 to check whether or not this limit cycle is locally stable.

We further compute from (15) that

$$W_{\alpha_1}=\begin{bmatrix}0.6043 & -0.3118 & 0.0248\\ 0 & 0 & 0\\ -0.1289 & -0.6862 & 0.5659\end{bmatrix},\qquad W_{\alpha_2}=\begin{bmatrix}0 & 0 & 0\\ 0.2119 & 0.8230 & 0.0363\\ -0.5397 & 0 & 0.7695\end{bmatrix}$$

$$W_{\beta_1}=\begin{bmatrix}0.8422 & -1.0944 & -0.0036\\ 0 & 0 & 0\\ -0.0897 & -1.5973 & 0.8768\end{bmatrix},\qquad W_{\beta_2}=\begin{bmatrix}0 & 0 & 0\\ 0.9701 & 0.5485 & 0.1463\\ 0.2804 & 0 & 0.5064\end{bmatrix}.$$

So,

$$\lambda(W_{\beta_2}W_{\beta_1}W_{\alpha_2}W_{\alpha_1})=\lambda\begin{bmatrix}0 & 0 & 0\\ -0.2129 & 0.0730 & 0.0192\\ -0.3261 & -0.0579 & 0.1582\end{bmatrix}=\{0.1421, 0.0891, 0\},$$

giving $\rho(W_{\beta_2}W_{\beta_1}W_{\alpha_2}W_{\alpha_1})=0.1421<1$. Hence, we conclude from Theorem 4.1 that the limit cycle is locally stable. Figure 3 shows the convergence of the two outputs for a trajectory starting from $[1\ 0.5\ 0]^T$. From the simulation, it is seen that the trajectory is convergent to the limit cycle.

6. Conclusion

This paper studies the local stability of limit cycles for MIMO systems under decentralized relay feedback. A sufficient condition is given based on the exact expression of the Poincare map. The result is an extension of the existing one for SISO relay feedback systems.

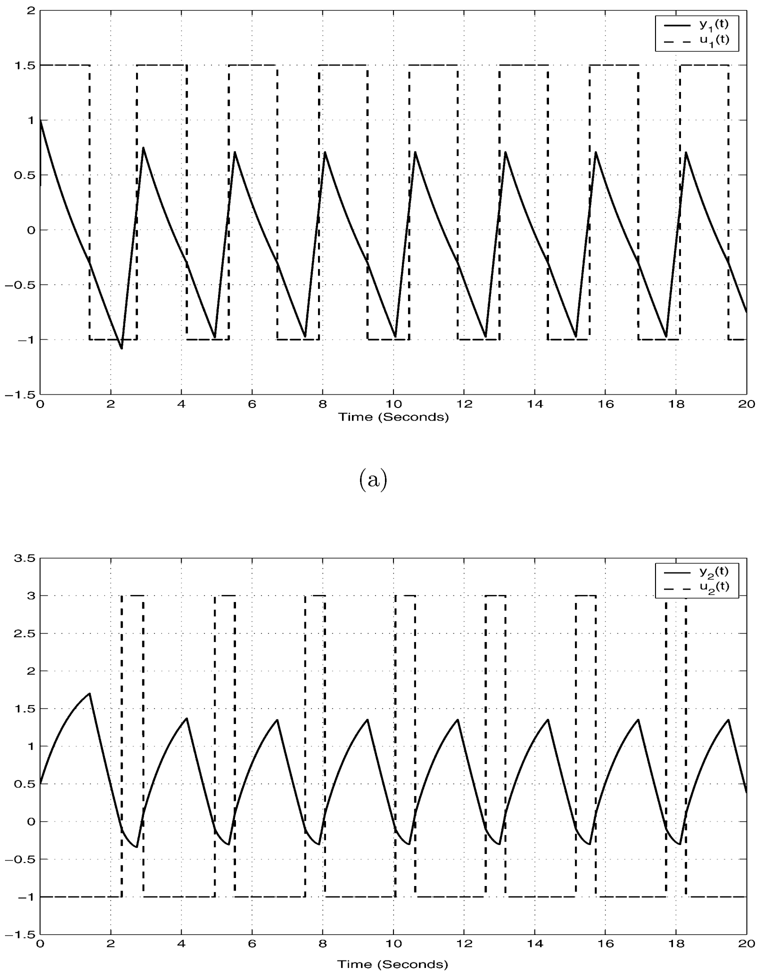

Fig. 3. Convergence process of the system in Example 5.1.

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